ISRAEL JOURNAL OF MATHEMATICS 109 (1999), 61–73

LIMIT THEOREMS FOR NON-HYPERBOLIC AUTOMORPHISMS OF THE TORUS

BY

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ABSTRACT

We prove the Donsker and Strassen invariance principles and other results for ergodic sums associated to regular functions for non-hyperbolic automorphisms of the torus. For this we use arithmetical and geometrical considerations which allow us to apply Gordin's method.

Introduction

In an ergodic dynamical system (X, T, μ), a function f on X defines a stationary process $(T^n f)_{n\geq 0}$. According to Birkhoff's theorem, if f is integrable, the law of large numbers holds for this process. To make this law precise it is sometimes possible to prove the central limit theorem (CLT) for the process $(T^n f)_{n\geq 0}$, that is to show there exists a positive real number $\sigma = \sigma(f) > 0$ such that

$$\lim_{n\to\infty} \sup_{\alpha} \left| \mu \left\{ x \ : \ \frac{1}{\sigma\sqrt{n}} \left(S_n f(x) - n \int f \, d\mu \right) < \alpha \right\} - \Phi(\alpha) \right| = 0,$$

where Φ is the distribution function of the reduced centered normal law and $S_n f$ is the sum $f + Tf + \cdots + T^{n-1} f$.

For example, in the forties, Fortet [F] and Kac [Ka] proved the CLT for regular functions for the endomorphisms of the circle. Later, Leonov [L1] obtained a similar result for every ergodic endomorphism of compact abelian groups. Here we address the problem of refinements of the CLT for regular functions in the case of ergodic automorphisms of the torus.

Let M be a square integer matrix of determinant ± 1. This matrix defines an automorphism of the d-dimensional torus,

$$T\colon \mathbb{T}^d \longrightarrow \mathbb{T}^d\colon x \longmapsto Mx \bmod 1,$$

Received March 30, 1997 and in revised form April 27, 1998

which leaves invariant the Lebesgue measure m. The dynamical system $(\mathbb{T}^d, T, m)$ is an ergodic dynamical system if and only if M has no eigenvalue which is a root of unity. From now on, T will denote an ergodic automorphism of the torus.

Obviously the result of Leonov applies to $(\mathbb{T}^d, T, m)$. We are interested in proving the following more precise stochastic limit properties.

THE CLT FOR SUBSEQUENCES. We say that f satisfies the CLT for subsequences if there exists $\sigma > 0$ such that, for every family $(k_n(x))_{n\geq 1}$ of increasing sequences in $\mathbb{N}$ such that, for almost all x, $\lim(k_n(x)/n) = c$, c being a constant, $0 < c < \infty$, the sequence $(1/\sqrt{n})S_{k_n(\cdot)}(\cdot)$ converges in law to $\mathcal{N}(0, \sigma^2/c)$.

THE FUNCTIONAL CLT. Let D be the set of right continuous functions with left limits on $[0,1]$ and $\mathcal{D}$ the Borel algebra on D for the Skorokhod topology. We say that the function f satisfies the functional CLT, if there exists $\sigma > 0$ such that the law of the process

$$\xi_n(t,x) = \frac{1}{\sigma\sqrt{n}} S_{[nt]}(x), \quad 0 \leq t \leq 1; \quad n = 1, 2, \ldots$$

converges weakly to the Wiener measure on $(D, \mathcal{D})$. The functional CLT is also called the Donsker invariance principle.

THE STRASSEN STRONG INVARIANCE PROPERTY. Let C be the space of the continuous functions on $[0,1]$ supplied with the uniform convergence topology and $\mathcal{K}$ be the set of the functions g absolutely continuous on $[0,1]$ such that

$$g(0) = 0 \quad \text{and} \quad \int \dot{g}^2\, dt \leq 1,$$

where $\dot{g}$ is the differential of g.

Let σ be a positive real number. For every x in X, consider the element $\zeta_n(\cdot, x)$ of C defined as the continuous piecewise affine interpolation of the application defined at k/n, $k \leq n$, by

$$\zeta_n(k/n, x) = \frac{1}{(2n\sigma^2 \log\log n)^{1/2}} S_k(x).$$

We say that the function f satisfies the Strassen strong invariance property if there exists σ such that, almost surely, the set $\{\zeta_n, n > 2\}$ is relatively compact in C and the set of its limit points coincides with $\mathcal{K}$. This is a strong version of the law of the iterated logarithm.

We remark that if there exists a measurable function h such that $f = h - Th$ (f is thus said to be a coboundary), then $S_n f/\sqrt{n}$ tends to zero in probability and the CLT does not hold.

When M has no modulus one eigenvalue, we say that T is hyperbolic. In this case $(\mathbb{T}^d, T)$ has Markov partitions and strong limit theorems can be proved for regular functions (see [GH]). There exist ergodic non-hyperbolic automorphisms of the torus. The matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

defines a 4-dimensional example. When T is non-hyperbolic, $(\mathbb{T}^d, T)$ has no Markov partition. So we have to use other tools to study the stochastic behaviour of ergodic sums. Thanks to the good distribution of the stable leaves of the automorphism, we can use Gordin's method to reduce the sequences $(T^n f)_{n\geq 0}$ to sequences of martingale differences. We deduce the following theorem.

THEOREM: *Let $f \in L^2(\mathbb{T}^d)$ with Fourier series*

$$f(\cdot) = \sum_{k\in\mathbb{Z}^d} c_k \exp(2i\pi\langle k, \cdot\rangle)$$

such that $c_0 = m(f) = 0$.

If there exist $R > 0$ and $\theta > 2$ such that, for every $b > 0$,

$$\text{(R)} \qquad \sum_{||k||>b} |c_k|^2 < R \log^{-\theta}(b),$$

then, if f is not a coboundary, f satisfies the CLT for subsequences, the functional CLT and the Strassen strong invariance principle.

The article is divided into four sections. In the first one we recall basic facts about martingale differences. In the second and third sections we prove a few preliminary results needed for the proof of the theorem, which is given in Section 4.

I. Martingale differences

To obtain the above stochastic properties several techniques are available in literature. For example, we could try to apply results of mixing theory ([IL], [OY]) or an operator method (see [GH]). Both work in the hyperbolic case but not in the non-hyperbolic case. So we shall use a third classical method: "martingale differences". Here we rapidly recall a few elements about this method (for a more detailed presentation see Hall and Heyde ([HH]) or Conze ([C])).

Definition I.1: A process defined by a sequence of random variables in $L^2(\mu)$, $(\ldots, X_{-1}, X_0, X_1, \ldots)$, is a sequence of **martingale differences** with respect to an increasing filtration $(\mathcal{F}_n)$ if it satisfies the following conditions:

(i) X_n is $\mathcal{F}_n$-measurable,
(ii) $\mathbb{E}(X_n|\mathcal{F}_{n-1}) = 0$, for every n.

Billingsley ([Bi]) and Ibragimov have proved the CLT for sequences of martingale differences. Since then, a lot of other results have been obtained. With the terminology of dynamical systems we can state the following theorem.

THEOREM I.2: *Let $(X, \mathcal{B}, \mu, T)$ be an invertible ergodic dynamical system, f a function in $L^2(\mu)$ and $\mathcal{A}$ a sub-σ-field of $\mathcal{B}$ such that*

(i) $\mathcal{A} \subset T\mathcal{A}$,
(ii) *f is $\mathcal{A}$-mesurable,*
(iii) $\mathbb{E}\left(f|T^{-1}\mathcal{A}\right) = 0$,
(iv) $\int f^2 d\mu = \sigma^2(f) > 0$.

Then the function f satisfies the functional CLT, the CLT for the subsequences and the Strassen strong invariance principle.

We can relax the hypothesis of this theorem.

Definition I.3: We say that a function f is a **coboundary** if there exists a measurable function h such that $f = h - Th$.

Definition I.4: We say that a function f is **homologous** to a function generating a sequence of martingale differences if f can be written in the form $f = g + h - Th$, where g is zero or satisfies the conditions of the preceding theorem.

As remarked by Gordin ([Go1]), a lot of stochastic properties of ergodic sums associated to a function f still hold for the functions which are homologous to f. This allows one to extend the above theorem. The following theorem, using a hilbertian homology criterion, can be found in [HH]:

THEOREM I.5: *Let $(X, \mathcal{B}, \mu, T)$ be an invertible ergodic dynamical system, $\mathcal{A}$ a subfield of $\mathcal{B}$ such that $\mathcal{A} \subset T\mathcal{A}$ and f a function in $L^2(\mu)$ such that*

$$\sum_{n>0} \|E(f|\mathcal{A}_n)\|_2 < \infty \quad \text{and} \quad \sum_{n<0} \|f - E(f|\mathcal{A}_n)\|_2 < \infty,$$

where $\mathcal{A}_n = T^{-n}\mathcal{A}$. Then, if f is not a coboundary, it satisfies the functional CLT, the CLT for the subsequences and the Strassen strong invariance principle. If f is a coboundary, then there exists a function h in $L^2(\mu)$ such that $f = h - Th$.

II. Partition and filtration

We recall that M is a square integer matrix **without an eigenvalue which is a root of unity,** and of determinant ± 1. This matrix necessarily has eigenvalues inside and outside the unit circle. Let F_u (resp. F_s, resp. F_e) be the M-stable vector space associated to the eigenvalues of modulus larger than (resp. smaller than, resp. equal to) 1. Let $v_1, \ldots, v_d$ be a basis of $\mathbb{R}^d$ in which M is represented by a real Jordan matrix. Let r denote the dimension of F_s and suppose that $v_1, \ldots, v_r$ is a basis of F_s. In $\mathbb{R}^d$ we fix the following norm: when $x = \sum_1^d x_i v_i$, take

$$||x|| = (\sum_{i=1}^{d} |x_i|^2)^{1/2}.$$

Let $B(x,\gamma)$ (resp. $B_i(x,\gamma)$) denote the ball centered at x and of radius γ in $\mathbb{R}^d$ (resp. F_i) ($i = e, u$ or s), m_s the Lebesgue measure on F_s associated to the basis $v_1, \ldots, v_r$ and $d(\cdot,\cdot)$ the distance induced by $||\cdot||$ on $\mathbb{R}^d$. Define

$$\rho = \frac{\min\{||k|| \;/\; k \in \mathbb{Z}^d \smallsetminus \{0\}\}}{2||M||}.$$

Consider a partition $\mathcal{P}$ of the torus whose elements are of diameter less than ρ and of the form $\sum I_i v_i$, where the I_i are intervals. (Such a partition is easily constructed. In $\mathbb{R}^d$, cover a compact fundamental domain of $\mathbb{T}^d$ with a finite number of sets of the requested form and diameter. Denote $\{R_i\}_{i\in I}$ the family of these sets and their image by all integer translations. This family gives a cover of $\mathbb{R}^d$. The family $\{D_j\}_{j\in\{0,1\}^I} = \{\bigcap_{i\in I} A_i / \forall_i A_i = R_i \text{ or } {}^c R_i\}$ gives a partition of $\mathbb{R}^d$ which is invariant by integer translations, that is a partition of $\mathbb{T}^d$. The D_j may not have the requested form but they are finite unions of sets with the right form.)

For two integers $k \le l$, let $\mathcal{P}_k^l$ denote the partition induced by $T^{-k}\mathcal{P}, \ldots, T^{-l}\mathcal{P}$ and $\mathcal{P}_k^l(x)$ denote the element of $\mathcal{P}_k^l$ containing x. For every couple of integers (k,l) the elements of $\mathcal{P}_k^l$ are convex sets $P_s^{kl} + P_u^{kl} + P_e^{kl}$ (each P_i^{kl} being included in a leaf $x + F_i$). For every integer k, let $\mathcal{P}_k^\infty(x)$ (resp. $\mathcal{P}_{-\infty}^k(x)$, resp. $\mathcal{P}_{-\infty}^\infty(x)$) be the set $\bigcap_{l=k}^\infty \mathcal{P}_k^l(x)$ (resp. $\bigcap_{l=-\infty}^k \mathcal{P}_l^k(x)$, resp. $\bigcap_{l=0}^\infty \mathcal{P}_{-l}^l(x)$).

When T is non-hyperbolic (i.e. M has modulus one eigenvalues), one can show that it is possible to find points x of the torus such that $\mathcal{P}_{-\infty}^\infty(x) \neq \{x\}$ ([Li] corollary 4.3). But, as was proved by Lind ([Li] theorem 1), the set of those points has Lebesgue measure zero. More precisely, he has proved that almost surely the set $\mathcal{P}_0^\infty(x)$ is included in $x+F_s$. In view of the choice of the partition $\mathcal{P}$ it is obvious that $\mathcal{P}_0^\infty(x)$ is a bounded convex set.

PROPOSITION II.1: *For almost every x, the local leaf $\mathcal{P}_0^\infty(x)$ is a bounded convex set included in $x+F_s$ with non-empty interior in $x+F_s$.*

Proof: The only assertion to prove is the one on the interior of $\mathcal{P}_0^\infty(x)$. The notations introduced in this proof will be used below in the proof of the theorem. Let η and $\epsilon \in]0,1[$ be such that, on F_s, M^{-n} expands distances with a multiplicative coefficient larger than $\eta\epsilon^{-n}$ and let β be in $]0,1[$. If we show that the measure of the set

$$V_n = \{x \ / \ \mathcal{P}_0^\infty(x) \text{ contains a ball in } x+F_s \text{ of radius } \eta\beta^n\}$$

tends to 1 when n tends to infinity, the proof of the proposition will be complete.

Consider the set

$$W_n = \left\{y \ / \ d\left(T^j y, \partial\mathcal{P}\left(T^j y\right)\right) \geq \epsilon^j\beta^n, \ \forall j \geq 0\right\}.$$

If $y \in W_n$, then $\mathcal{P}\left(T^j y\right) \cap \left(T^j y + F_s\right)$ contains the ball $B_s\left(T^j y, \beta^n\epsilon^j\right)$. So the set $T^{-j}(\mathcal{P}\left(T^j y\right) \cap \left(T^j y + F_s\right))$ contains the ball $B_s\left(y, \beta^n\eta\right)$. But we have

$$\mathcal{P}_0^\infty(y) \supset \cap_{j\geq 0} T^{-j}\left(\mathcal{P}\left(T^j y\right) \cap \left(T^j y + F_s\right)\right).$$

Hence, if $y \in W_n$, $\mathcal{P}_0^\infty(y)$ contains the ball $B_s\left(y, \beta^n\eta\right)$. In other words, V_n contains W_n. On the other hand, in view of the form of the boundaries of the elements of $\mathcal{P}$ and thanks to the T-invariance of m, one proves that there exists $L>0$ such that

$$m\{y \ / \ d(T^j y, \partial\mathcal{P}(T^j y)) \leq \epsilon^j\beta^n\} \leq L\epsilon^j\beta^n.$$

Hence the set ${}^cW_n$ has measure less than $\sum_{j\geq 0} L\epsilon^j\beta^n = L(1-\epsilon)^{-1}\beta^n$ and ${}^cV_n$ has the same property. ∎

Now define $\mathcal{A}$ as the σ-field generated by the partition $\mathcal{P}_0^\infty$ and $\mathcal{A}_n$ as $T^{-n}\mathcal{A}$. Then $\mathcal{A}_n$ is the algebra generated by $\mathcal{P}_n^\infty$ and the sequence $(\mathcal{A}_n)_{n\in\mathbb{Z}}$ is a decreasing filtration. The partition $\mathcal{P}_n^\infty$ is measurable in the sense of Rokhlin ([R]). So, for almost every element $\mathcal{P}_n^\infty(x)$ of $\mathcal{P}_n^\infty$, we can define a conditional probability $m_{\mathcal{P}_n^\infty(x)}$ on $\mathcal{P}_n^\infty(x)$ such that, for m-almost every x, one has

$$E\left(f|\mathcal{A}_n\right)(x) = \int_{\mathcal{P}_n^\infty(x)} f(t) \ dm_{\mathcal{P}_n^\infty(x)}(t).$$

LEMMA II.2: *Let x be such that $\mathcal{P}_n^\infty(x)$ is a set included in $x+F_s$ with non-empty interior. Let m_s be the Lebesgue measure on F_s associated to the basis $v_1,\dots,v_r$. Then we have*

$$m_{\mathcal{P}_n^\infty(x)} = \frac{m_s|_{\mathcal{P}_n^\infty(x)}}{m_s\left(\mathcal{P}_n^\infty(x)\right)}.$$

For almost every x, the local leaf $\mathcal{P}_n^\infty(x)$ is a set included in $x+F_s$ with non-empty interior. So almost surely we have (note that $\mathcal{P}_n^\infty = T^{-n}\mathcal{P}_0^\infty$)

$$\begin{aligned} E(f|\mathcal{A}_n)(x) &= \int_{\mathcal{P}_n^\infty(x)} f(t)\,dm_{\mathcal{P}_n^\infty(x)}(t) \\ &= \frac{1}{m_s\left(\mathcal{P}_n^\infty(x)\right)}\int_{\mathcal{P}_n^\infty(x)} f(t)\,dm_s(t) \\ &= \frac{1}{m_s\left(T^{-n}\mathcal{P}_0^\infty(T^n x)\right)}\int_{T^{-n}\mathcal{P}_0^\infty(T^n x)} f(t)\,dm_s(t). \end{aligned}$$

III. The distribution of the contracting leaves

As we mentionned in the introduction, we shall use the good distribution of the contracting leaves in the torus. This can be expressed by Proposition III.3 below.

The following two lemmas will be used in the proof of Proposition III.3. The first one is taken from [K] or [L2].

LEMMA III.1: *Let M be a $d\times d$ matrix with integral coefficients. Let V and V_1 be two M-stable subspaces of $\mathbb{R}^d$ such that $\mathbb{R}^d = V\bigoplus V_1$ and the restrictions of M to V and V_1 do not have common eigenvalues. If $V\cap\mathbb{Z}^d = 0$, there exists a constant $K>0$ such that, for every $k\in\mathbb{Z}^d\setminus\{0\}$, one has*

$$d(k,V)\geq K||k||^{-q},$$

where q denotes the dimension of V.

Let C be a compact convex set in F_s. Let $a(C)$ denote the area of the boundary of C, that is the $(r-1)$-dimensional measure of this boundary computed in basis $v_1,\dots,v_r$.

LEMMA III.1: *Let C be a convex compact set included in $x+F_s$ with non-empty interior (in F_s) and γ be a positive real number. We have the following inequality:*

$$m_s\{y\in C\ /\ d(y,\partial C)\leq\gamma\}\leq\gamma a(C).$$

PROPOSITION III.3: *There exist $K > 0$ and $\lambda \in]0,1[$ such that, for every x, every convex compact C included in $x + F_s$ with non-empty interior (in F_s) and every $k \in \mathbb{Z}^d \setminus \{0\}$, we have*

$$\frac{1}{m_s(T^{-n}C)}\left|\int_{T^{-n}C} \exp(2i\pi\langle k,t\rangle)\ dm_s(t)\right| < K\frac{a(C)}{m_s(C)}\|k\|^{d-r}\lambda^n.$$

Proof: Fix an element k of $\mathbb{Z}^d \setminus \{0\}$. There exist a constant K_1 independent of k and $v_1^k, \ldots, v_r^k$ a basis isometric to $v_1, \ldots, v_r$ such that for every j we have

$$|\langle k, v_j^k\rangle|^{-1} \leq K_1\|\Pi_s k\|^{-1},$$

where Π_s denotes the orthogonal (for the canonical coordinates) projection on F_s. For every family of r integers $(l_1, \ldots, l_r)$, define the set $F^k_{l_1,\ldots,l_r}$ by

$$F^k_{l_1,\ldots,l_r} = \left\{\sum_1^r x_j\langle k, v_j^k\rangle^{-1} v_j^k \ / \ l_j \leq x_j \leq l_j + 1,\ \forall j = 1, \ldots, r\right\}.$$

We have

$$\int_{F^k_{l_1,\ldots,l_r}} \exp(2i\pi\langle k,t\rangle)\ dm_s(t) = \prod_{j=1}^r \int_{l_j}^{l_j+1} \exp(2i\pi\langle k, v_j^k\rangle^{-1}\langle k, v_j^k\rangle x_j)\ dx_j$$
$$=0.$$

Thus, if D_n^k denotes the set of the points $T^{-n}C$ which do not belong to the union of the parallelepipeds $F^k_{l_1,\ldots,l_r}$ included in $T^{-n}C$, we immediately obtain the inequality

$$\left|\int_{T^{-n}C} \exp(2i\pi\langle k,t\rangle)\ dm_s(t)\right| \leq m_s\left(D_n^k\right).$$

There exists K' such that, if $y \in D_n^k$, then we have

$$d(y, \partial T^{-n}C) \leq K' \max_j |\langle k, v_j^k\rangle|^{-1};$$

hence

$$D_n^k \subset \left\{y \in T^{-n}C \ / \ d\left(y, \partial T^{-n}C\right) \leq K_1K'\|\Pi_s k\|^{-1}\right\}.$$

On the other hand, the restriction of M to F_s is contracting. So there exist $K_2 > 0$ and $\lambda \in]0,1[$ such that

$$T^n D_n^k \subset \left\{y \in C \ / \ d(y, \partial C) \leq K_2\lambda^n\|\Pi_s k\|^{-1}\right\}.$$

According to Lemma III.2, we have

$$m_s(T^n D_n^k) \leq K_2 \lambda^n ||\Pi_s k||^{-1} a(C);$$

then

$$m_s(D_n^k) \leq K_2 \lambda^n ||\Pi_s k||^{-1} \Delta^n a(C),$$

where Δ denotes the absolute value of the determinant of the restriction of M^{-1} to F_s. It follows (note that $m_s(T^{-n}C) = \Delta^n m_s(C)$) that

$$\frac{m_s\left(D_n^k\right)}{m_s\left(T^{-n}C\right)} \leq K_2 \lambda^n ||\Pi_s k||^{-1} \frac{a\left(C\right)}{m_s\left(C\right)}.$$

We conclude by using the inequality given by Lemma III.1: there exist K_3 and K_4 such that

$$||\Pi_s k||^{-1} \leq K_3 d(k, F_u + F_e)^{-1} \leq K_4 ||k||^{d-r}. \qquad \blacksquare$$

IV. Proof of the theorem

Let $f \in L^2(\mathbb{T}^d)$ satisfying the condition (R) of the Theorem. According to Theorem I.5, to prove the theorem it suffices to show that the two series

$$\sum_{n<0} ||f - E\left(f|\mathcal{A}_n\right)||_2 \quad \text{and} \quad \sum_{n>0} ||E\left(f|\mathcal{A}_n\right)||_2$$

converge.

Let $b(n)$ be a sequence to be defined later. Define

$$f_{1n} = \sum_{||k|| \leq b(n)} c_k \exp(2i\pi\langle k, \cdot\rangle) \quad \text{and} \quad f_{2n} = \sum_{||k|| > b(n)} c_k \exp(2i\pi\langle k, \cdot\rangle).$$

The atoms of $\mathcal{A}_n$ are almost surely pieces of contracting leaves. Hence there exist real numbers $Q > 0$ and $\delta \in]0, 1[$ such that almost surely the diameter of $\mathcal{P}_n^\infty(x)$ is less than $Q\delta^{-n}$ when n is negative. So there exists $L_0 > 0$ such that the following inequalities hold:

$$\begin{aligned} ||f_{1n} - E\left(f_{1n}|\mathcal{A}_n\right)||_2 \leq & ||f_{1n} - E\left(f_{1n}|\mathcal{A}_n\right)||_\infty \\ \leq & \sum_{||k||<b(n)} |c_k| \; ||\exp(2i\pi\langle k, \cdot\rangle) - \mathbb{E}(\exp(2i\pi\langle k, \cdot\rangle)|\mathcal{A}_n)||_\infty \\ \leq & \sum_{||k||<b(n)} |c_k| \; 2\pi \; ||k|| \; Q\delta^{-n} \\ \leq & L_0 b(n)^{d+1} \delta^{-n}. \end{aligned}$$

On the other hand, we have (hypothesis (R))

$$||f_{2n} - E\left(f_{2n}|\mathcal{A}_n\right)||_2 \le 2||f_{2n}||_2 \le 2R^{1/2}\log^{-\theta/2}(b(n)).$$

As

$$||f - E\left(f|\mathcal{A}_n\right)||_2 \le ||f_{1n} - E\left(f_{1n}|\mathcal{A}_n\right)||_2 + ||f_{2n} - E\left(f_{2n}|\mathcal{A}_n\right)||_2,$$

by choosing $b(n) = \delta^{n/2(d+1)}$, we see that the series $\sum_{n<0} ||f - E\left(f|\mathcal{A}_n\right)||_2$ converges.

Let $b'(n)$ be a sequence to be defined later. Define

$$f'_{1n} = \sum_{||k||\le b'(n)} c_k \exp(2i\pi\langle k,\cdot\rangle) \quad \text{and} \quad f'_{2n} = \sum_{||k||>b'(n)} c_k \exp(2i\pi\langle k,\cdot\rangle).$$

According to Proposition III.3 there exists K such that, almost surely,

$$\begin{aligned} &|\mathbb{E}(\exp\left(2i\pi\langle k,\cdot\rangle\right)|\mathcal{A}_n)(x)| \\ &= \frac{1}{m_s\left(T^{-n}\mathcal{P}_0^\infty(T^n x)\right)}\left|\int_{T^{-n}\mathcal{P}_0^\infty(T^n x)} \exp\left(2i\pi\langle k,t\rangle\right)\, dm_s\left(t\right)\right| \\ &\le K\frac{a\left(\mathcal{P}_0^\infty(T^n x)\right)}{m_s\left(\mathcal{P}_0^\infty(T^n x)\right)}||k||^{d-r}\lambda^n. \end{aligned}$$

Let β be such that $\beta^r \in]\lambda, 1[$, η as in Proposition II.1 and consider the set

$$V'_n = \left\{x \;/\; \mathcal{P}_0^\infty\left(T^{-n}x\right) \text{ contains a ball in } T^{-n}x + F_s \text{ of radius } \eta\beta^n\right\}.$$

When x belongs to V'_n, $\mathcal{P}_0^\infty(T^n x)$ contains a ball of radius β^n, thus $m_s(\mathcal{P}_0^\infty(T^n x)) \ge R_0\beta^{rn}$ for some constant R_0. Thus there exists a real number $R_1 > 0$ such that

$$\begin{aligned} \int_{V'_n} |E\left(f'_{1n}|\mathcal{A}_n\right)|^2\, dm &= \int_{V'_n}\left|\sum_{||k||<b'(n)} c_k\mathbb{E}(\exp(2i\pi\langle k,\cdot\rangle)|\mathcal{A}_n)(x)\right|^2 dm(x) \\ &\le \sup_{x\in V'_n}\left(\sum_{||k||<b'(n)} |c_k| K\frac{a\left(\mathcal{P}_0^\infty(T^n x)\right)}{m_s\left(\mathcal{P}_0^\infty(T^n x)\right)}\, ||k||^{d-r}\, \lambda^n\right)^2 \\ &\le R_1 b'(n)^{2(2d-r)}(\lambda/\beta^r)^{2n}. \end{aligned}$$

(The numbers $a\left(\mathcal{P}_0^\infty(T^n x)\right)$ are uniformly bounded.)

We have already seen that $m({}^cV'_n) \leq L(1-\epsilon)^{-1}\beta^n$ for a real number $L > 0$ (Proposition II.1); so there exists $R_2 > 0$ such that

$$\begin{aligned}\int_{{}^cV'_n} |E(f'_{1n}|\mathcal{A}_n)|^2 \, dm \leq & m({}^cV'_n) \left(\sum_{||k||<b'(n)} |c_k| \right)^2 \\ \leq & R_2 b'(n)^{2d} \beta^n.\end{aligned}$$

Now we can write

$$\begin{aligned}||E(f'_{1n}|\mathcal{A}_n)||_2 \leq & (\int_{V_n} |E(f'_{1n}|\mathcal{A}_n)|^2 \, dm + \int_{{}^cV_n} |E(f'_{1n}|\mathcal{A}_n)|^2 \, dm)^{1/2} \\ \leq & \left(R_1 b'(n)^{2(2d-r)} (\lambda/\beta^r)^{2n} + R_2 b'(n)^{2d} \beta^n \right)^{1/2} \\ \leq & R_3 b'(n)^{2d-r} \gamma^n,\end{aligned}$$

for some $R_3 > 0$ and $\gamma \in]0,1[$. Obviously we have (hypothesis (R))

$$||E(f'_{2n}|\mathcal{A}_n)||_2 \leq ||f'_{2n}||_2 \leq R^{1/2} \log^{-\theta/2}(b'(n)),$$

and, as

$$||E(f|\mathcal{A}_n)||_2 \leq ||E(f'_{1n}|\mathcal{A}_n)||_2 + ||E(f'_{2n}|\mathcal{A}_n)||_2,$$

by choosing $b'(n) = \gamma^{-n/2(2d-r)}$, we see that the series $\sum_{n>0} ||E(f|\mathcal{A}_n)||_2$ converges. ∎

Remark 1: The condition (R) of the theorem is satisfied under the following decreasing property of Fourier coefficients of f:

$$|c_{(k_1,\ldots,k_d)}| \leq A \prod_1^d \frac{1}{(1+|k_i|)^{1/2} \log^{\theta}(2+|k_i|)},$$

with $A > 0$ and $\theta > 2$. This is a slightly stronger hypothesis than the one obtained by Leonov. We recall that Leonov proved the CLT (possibly degenerated) under the analogous hypothesis with $\theta > 3/2$.

Remark 2: Let $\mathcal{B}_n$ denote the σ-field generated by $\mathcal{P}_{-\infty}^{-n}$. In the non-hyperbolic case, by adapting a proof of Lind, we can show that Rosenblatt's coefficient

$$\sup_{A \in \mathcal{A}_0,\ B \in \mathcal{B}_n} |m(A \cap B) - m(A)\, m(B)|$$

does not tend to zero when n tends to infinity (see [Lb]).

Remark 3: To apply the theorem, it is important to know whether f is a coboundary or not. Let f be a Hölder continuous function whose Fourier series is absolutely convergent. Livshits has shown ([Lv] remark 3) that if f is a coboundary in $L^1(\mathbb{T}^d)$, then f sums to zero on periodic orbits of T. Such a function f satisfies the condition (R) of the theorem. Thus, if f is a measurable coboundary, it is a coboundary in $L^2(\mathbb{T}^d)$ (see Theorem I.5.). So if there exists a periodic orbit on which f doesn't sum to zero, then f is not a coboundary and satisfies the mentioned limit theorems.

Remark 4: For a hyperbolic automorphism, it is shown in [Lv] that, if a Hölder continuous function whose Fourier series is absolutely convergent is a coboundary in $L^1(\mathbb{T}^d)$, then it is a coboundary in the space of continuous functions. For an ergodic automorphism, thanks to remark 3 of [Lv], a result of Veech ([V] corollary 5.14) and our theorem, we can state: if a d times differentiable function with a Hölder continuous d^{th} differential is a measurable coboundary, then it is a coboundary in the space of Hölder continuous functions.

Remark 5: For indicator functions of regular sets, the same limit theorems can be proved by using another method inspired by Katznelson's proof of bernoullicity of ergodic automorphisms of the torus (see [Lb]).

ACKNOWLEDGEMENT: The author is grateful to Jean-Pierre Conze who supervised this work, and to the referee for useful remarks.

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